1. Evaluate $\int \frac{x}{x^2 - 2x - 3}\, dx$ using Partial Fractions. Follow the steps below.

   (a) Is the degree of the numerator greater than (or equal to) the degree of the denominator for the function in the integrand?
   If yes, then use long division before proceeding to utilize the method of partial fractions. If no, continue this problem without doing long division.

   Solution:
   No, the degree of the numerator is not greater than the degree of the denominator.

   (b) Set up the method of partial fractions for your rational function. Make sure to factor out your denominator completely beforehand.

   Solution:
   $$\frac{x-7}{x^2-2x-3} = \frac{x-7}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

   (c) Write and solve a system of equations for your unknown constants by equating coefficients of like terms and by equating constants.

   Solution:
   Multiplying both sides of the equation in (b) by $(x-3)(x+1)$,
   $$\begin{aligned} x - 7 &= A(x+1) + B(x-3) \\ &= (A+B)x + A - 3B. \end{aligned}$$

   Hence, we end up solving the following system of equations:
   $$\begin{aligned} A + B &= 1 \\ A - 3B &= -7 \end{aligned}$$

   Subtracting the first equation from the second equation, we get that $-4B = -8$, so $B = 2$. Thus, $A = -1$.

   (d) Evaluate your integral using the partial fraction representation of your rational function.

   Solution:
   $$\int \frac{x-7}{x^2-2x-3}\, dx = \int \frac{-1}{x-3} + \frac{2}{x+1}\, dx = -ln|x-3| + 2ln|x+1| + C$$

2. Evaluate $\displaystyle\int \frac{3x^4+1}{(x^2+1)(x-1)}\,dx$ by using Partial Fractions.

Solution:

The degree of the numerator is not greater than the degree of the denominator, so we will have to perform a long division before we can use a partial fraction decomposition to simplify the integral. This looks like

$$\begin{array}{r|l} & 3x+3, \\ x^3-x^2+x-1 & 3x^4 \qquad\qquad\qquad +1 \\ & -3x^4+3x^3-3x^2+3x \\ \hline & 3x^3-3x^2+3x+1 \\ & -3x^3+3x^2-3x+3 \\ \hline & 4 \end{array}$$

so we have

$$\int \frac{3x^4+1}{(x^2+1)(x-1)}\,dx = \int 3x+3+\frac{4}{(x^2+1)(x-1)}\,dx.$$

To split the fraction, we begin with the general form

$$\frac{4}{(x^2+1)(x-1)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+1},$$

and we multiply by $(x^2+1)(x-1)$ to get

$$4 = A(x^2+1)+(Bx+C)(x-1).$$

Now, in the case that $x=1$, we find $2A=4$ or

$$A=2.$$

Now let's just choose relatively simple x values such as $x=0$ and $x=2$. These give us

$$4 = 2(0^2+1)+C(-1),$$
$$4 = 2(2^2+1)+(2B+C)(1),$$

or

$$B=-2 \qquad \text{and} \qquad C=-2.$$

Thus,

$$\frac{4}{(x^2+1)(x-1)} = \frac{2}{x-1}-\frac{2x+2}{x^2+1},$$

and we can find

$$\begin{aligned}\int \frac{3x^4+1}{(x^2+1)(x-1)}\,dx &= \int 3x+3+\frac{2}{x-1}-\frac{2x+2}{x^2+1}\,dx \\ &= \int 3x+3+\frac{2}{x-1}-\frac{2x}{x^2+1}-\frac{2}{x^2+1}\,dx \\ &= \frac{3}{2}x^2+3x+2\ln|x-1|-\ln|x^2+1|-2\arctan(x)+C.\end{aligned}$$

3. Evaluate $\int \frac{3\cos(x)}{\sin^2(x)+\sin(x)}\,dx$ by first making a substitution, and then using Partial Fractions.

Solution:

As the problem suggests, we begin by setting $u = \sin(x)$ to get

$$\int \frac{3\cos(x)}{\sin^2(x)+\sin(x)}\,dx = \int \frac{3}{u^2+u}\,du$$

Now, we don't know how to take this integral, but we can use partial fractions!

$$\frac{3}{u^2+u} = \frac{3}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$$

or

$$3 = A(u+1) + Bu.$$

Setting $u = 0$ and $u = -1$ gives us

$$A = 3 \qquad \text{and} \qquad B = -3,$$

so

$$\frac{3}{u^2+u} = \frac{3}{u} - \frac{3}{u+1}.$$

Hence,

$$\begin{aligned}
\int \frac{3\cos(x)}{\sin^2(x)+\sin(x)}\,dx &= \int \frac{3}{u^2+u}\,du \\
&= \int \frac{3}{u} - \frac{3}{u+1}\,du \\
&= 3\ln|u| - 3\ln|u+1| + C \\
&= 3\ln|\sin(x)| - 3\ln|\sin(x)+1| + C.
\end{aligned}$$

4. Evaluate $\displaystyle\int \frac{7e^{2x}}{e^{2x}-2e^x-3}\,dx$ by first making a substitution, and then use Partial Fractions. Hint: $e^{2x} = e^x \cdot e^x$.

Solution:

Once again, we begin by setting $u = e^x$ to get

$$\int \frac{7e^{2x}}{e^{2x}-2e^x-3}\,dx = \int \frac{7u}{u^2-2u-3}\,du$$

And again, we can use partial fractions.

$$\frac{7u}{u^2-2u-3} = \frac{7u}{(u-3)(u+1)} = \frac{A}{u-3} + \frac{B}{u+1}$$

or

$$7u = A(u+1) + B(u-3).$$

Setting $u = 3$ and $u = -1$ gives us

$$A = \frac{21}{4} \qquad \text{and} \qquad B = \frac{7}{4},$$

so

$$\frac{7u}{u^2+u} = \frac{21/4}{u-3} + \frac{7/4}{u+1}.$$

Hence,

$$\begin{aligned}
\int \frac{7e^{2x}}{e^{2x}-2e^x-3}\,dx &= \int \frac{7u}{u^2-2u-3}\,du \\
&= \int \frac{21/4}{u-3} + \frac{7/4}{u+1}\,du \\
&= \frac{21}{4}\ln|u-3| + \frac{7}{4}\ln|u+1| + C \\
&= \frac{21}{4}\ln|e^x-3| + \frac{7}{4}\ln|e^x+1| + C.
\end{aligned}$$